Gravitation II

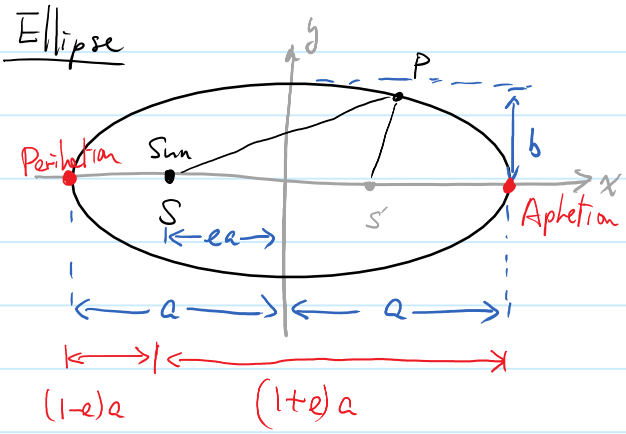
Kepler's Law (purely empirical laws)

- become evidence of Newton's Law of gravity.

Kepler 1st Law

Each planet moves in an elliptical orbit with the sun located at one of the foci of the ellipse.

Ellipse



$|PS| + |PS'| = \text{constant}.$

S, S' : foci

a : semi-major axis

b : semi-minor axis

e : eccentricity, $0 \le e < 1$

$e = 0 \rightarrow$ circle

useful formula: $e^2 = 1 - \frac{b^2}{a^2}$

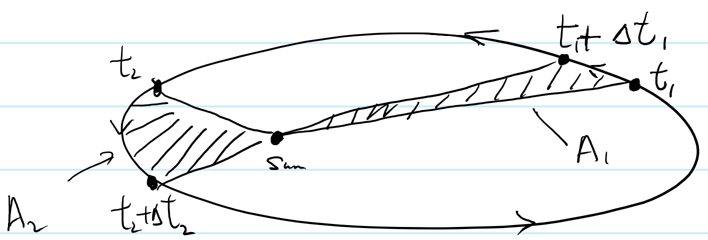
Perihelion : closest point from the Sun : $(1-e)a$

Aphelion : farthest point from the Sun : $(1+e)a$.

2nd Law

Line from the Sun to a planet sweeps out equal area in equal time interval.

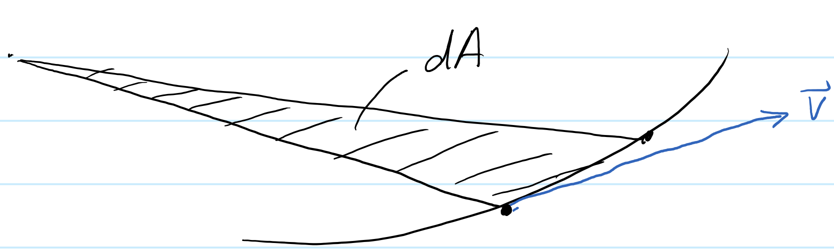
$\boxed{\text{Equal area, equal time}}$



if $\Delta t_1 = \Delta t_2$

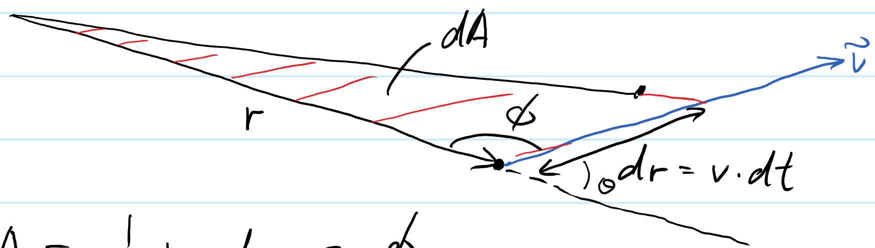
then $A_1 = A_2$

proof.



in time dt, the line sweeps an area dA.

Approximate dA by triangle



$\theta = \pi - \phi$

$$dA = \frac{1}{2} r \cdot dr \cdot \sin\phi$$

$$= \frac{1}{2} r\, v dt\, \sin(\pi - \theta)$$

$$= \frac{1}{2} r\, v\, dt\, \sin\theta$$

$$= \frac{1}{2} dt\, |\vec{r} \times \vec{v}|$$

$$= \frac{1}{2} \frac{dt}{m} |\vec{r} \times \vec{p}|$$

$$= \frac{1}{2} \frac{|\vec{L}|}{m} dt$$

$\vec{L}$ = angular momentum of the planet about the Sun = constant

$$dA = (\text{constant}) \cdot dt$$

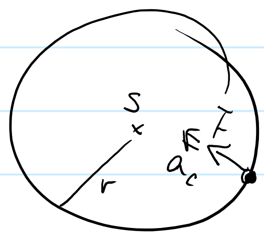
$\because \vec{\tau} = \vec{r} \times \vec{F}_g = \vec{0}$

$\Rightarrow \frac{d\vec{L}}{dt} = \vec{\tau} = \vec{0}$

3^{rd} Law

$$\text{Period}, T \propto a^{3/2}$$

for circular orbit. $(e=0)$



$$\frac{GMm}{r^2} = F = ma_c = m\frac{v^2}{r}$$

$$v = \frac{2\pi r}{T}$$

$$\Rightarrow \frac{GM}{r} = \frac{(2\pi)^2 r^2}{T^2}$$

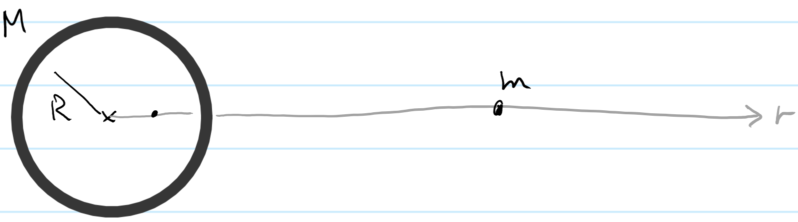
$$\Rightarrow \underline{T = \frac{2\pi}{\sqrt{GM}} r^{3/2}} \checkmark$$

True for all planets orbiting about the same Sun with mass M.

Spherical mass distribution

Shell theorem

For a spherical shell of mass M.

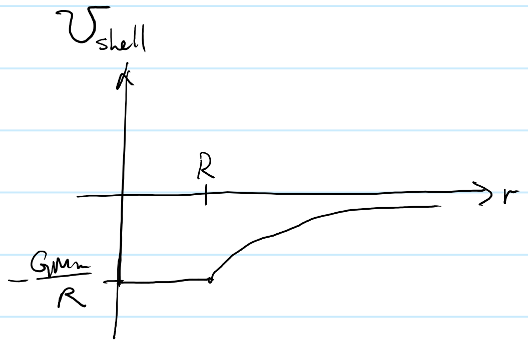


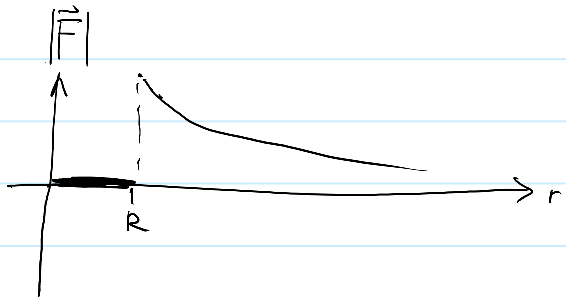
m outside of R, $r > R$ $\quad U(r) = -\dfrac{GMm}{r}$ $\quad$ same as two point masses.

m inside of R, $r \leq R$ $\quad U(r) = -\dfrac{GMm}{R}$

constant

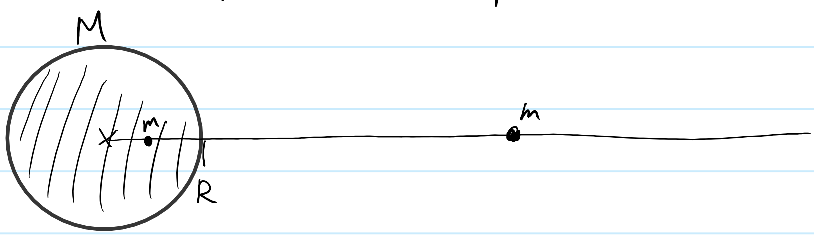
$$\Rightarrow \begin{cases} r > R & |\vec{F_g}| = \dfrac{GMm}{r^2} \\ r < R & |\vec{F_g}| = 0 \end{cases} \qquad F_r = -\frac{dU}{dr}$$





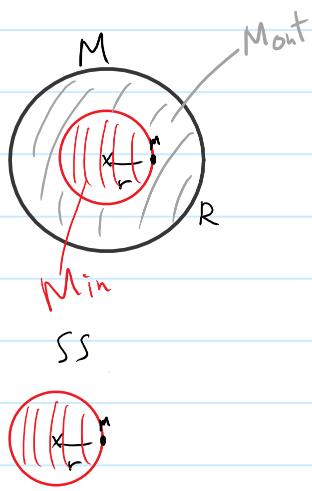
Application

Consider a uniform solid sphere.



for $r > R$,
$$\begin{cases} F = \dfrac{GMm}{r^2} \\ U = -\dfrac{GMm}{r}. \end{cases}$$

for $r < R$,

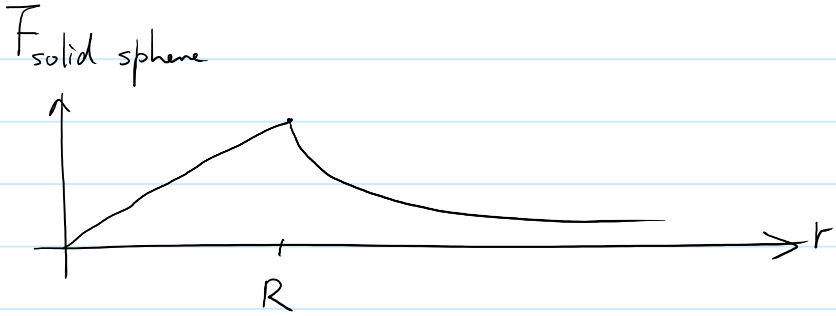


The mass in the gray region does not contribute any net force to m.
$\because$ m is inside the shells of M_{out}.

F_g by M_{in} is

$$F_g = \frac{GM_{in}m}{r^2} = \frac{G}{r^2}\frac{M}{\frac{4}{3}\pi R^3} \times \frac{4}{3}\pi r^3 \; m$$

$$= \frac{GMm}{R^3} r. \propto r$$

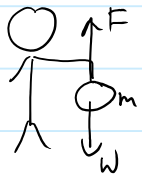


Apparent weight on Earth

Weight $\overset{def.}{=}$ gravitational force by the planet

$$W = \frac{G M_E m}{R_E^2}$$

Recall How do we know the weight?

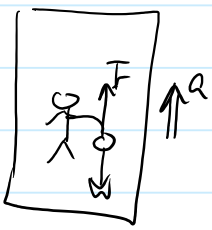


$$F - W = ma = 0$$

$$\Rightarrow \underline{F} = W = m\frac{GM}{R_E^2} = m g_0 \qquad g_0 = 9.8\ m/s^2$$

we measure this not W directly.

In non-inertial frame $\vec{a} \neq 0$



$$F - W = ma \neq 0$$

$$F = ma + W$$

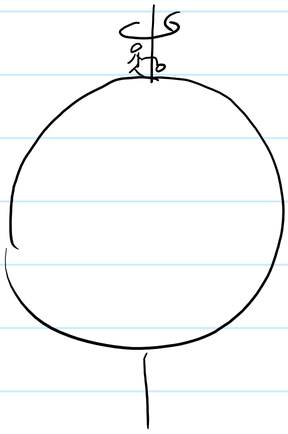
$$= ma + mg_0$$

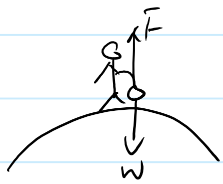
we will think F is the weight of the mass since it is the force to hold the mass. $\Rightarrow$ $F = W_{apparent}$.

$$F = m(g_0 + a) = m g_{apparent}$$

$$g_{apparent} = g_0 + a$$

At the pole (e.g North pole)

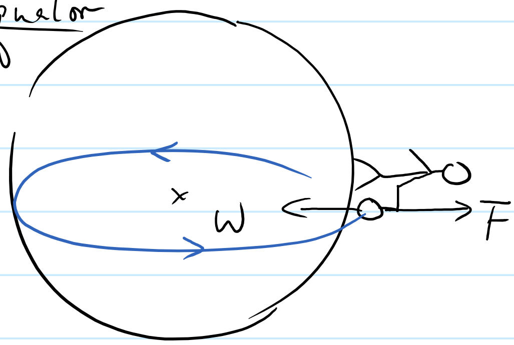




$$F - W = ma = 0$$

$$W_{apparent} = F = W = mg_0.$$

On Equator



The mass is rotating with the Earth under a uniform circular motion.

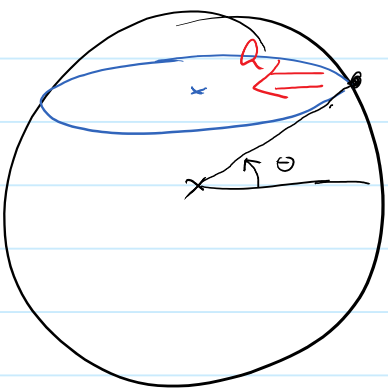
$$a = \frac{V^2}{R_E} = R_E \omega^2$$

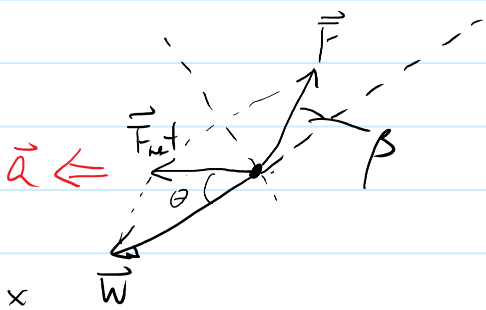
$$\omega = 2\pi / 24hr.$$

$$W - F = ma = mR_E\omega^2$$

$$W_{apparent} = mg_0 - mR_E\omega^2 = m(g_0 - R_E\omega^2) < mg_0$$

At elsewhere.





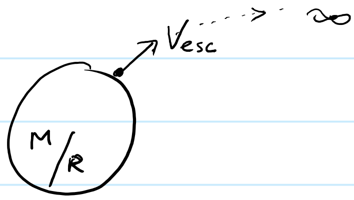
$$\vec{F}_{net} = m\vec{a}.$$

$\vec{F}$ cannot point to opposite direction of $\vec{W}$.

Answer: $\tan\beta = \frac{1}{2} \frac{\sin 2\theta}{(g_0/R_E\omega^2) - \cos^2\theta}$, If $\theta = 22.4°$ ~@HK
$\beta = 0.07°$

Black Hole

Recall: Escape velocity, $V_{esc} = \sqrt{\frac{2GM}{R}}$



Consider a collapsing star. $R \downarrow$ M is fixed, $V_{esc} \uparrow$

When V_{esc} increases to the value of the speed of light, c, not even light could escape the gravitational force of the star.

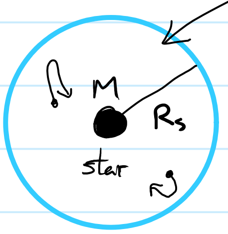
The radius of the star, R_s, such that $V_{esc} = c$ is

$$c = V_{esc} = \sqrt{\frac{2GM}{R_s}} \Rightarrow \boxed{R_s = \frac{2GM}{c^2}}$$

Schwarzchild Radius
(actually not derived by Newton's Law of gravity, but by General relativity)

The actual radius of the star could be less than R_s.

But any thing at a distance closer than R_s from the center of the star could not escape.



nothing inside this boundry can reach to the world outside.

$\Rightarrow$ It is apparently black to anyone look at it from the outside.